A note on interval edge-colorings of graphs

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An edge-coloring of a graph G with colors $1, 2, \ldots, t$ is called an interval t-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of G colored by i, and the colors of edges incident to any vertex of G are distinct and form an interval of integers. In this paper we prove that if a connected graph G with n vertices admits an interval t-coloring, then $t \leq 2n - 3$. We also show that if G is a connected r-regular graph with n vertices has an interval t-coloring and $n \geq 2r+2$, then this upper bound can be improved to $2n-5$.

Keywords: edge-coloring, interval coloring, bipartite graph, regular graph

1. Introduction

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. An (a, b)-biregular bipartite graph G is a bipartite graph G with the vertices in one part all having degree a and the vertices in the other part all having degree b. A partial edge-coloring of G is a coloring of some of the edges of G such that no two adjacent edges receive the same color. If α is a partial edge-coloring of G and $v \in V(G)$, then $S(v, \alpha)$ denotes the set of colors of colored edges incident to v.

An edge-coloring of a graph G with colors $1, 2, \ldots, t$ is called an interval t-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of G colored by i, and the colors of edges incident to any vertex of G are distinct and form an interval of integers. A graph G is interval colorable, if there is $t \geq 1$ for which G has an interval t-coloring. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the greatest value of t for which G has an interval t-coloring is denoted by $W(G)$.

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The concept of interval edge-coloring was introduced by Asratian and Kamalian [2]. In [2, 3] they proved the following theorem.

Theorem 1 *If G is a connected triangle-free graph and $G \in \mathfrak{N}$, then*

$$W(G) \leq |V(G)| - 1.$$

In particular, from this result it follows that if G is a connected bipartite graph and $G \in \mathfrak{N}$, then $W(G) \leq |V(G)| - 1$. It is worth noting that for some families of bipartite graphs this upper bound can be improved. For example, in [1] Asratian and Casselgren proved the following

Theorem 2 *If G is a connected (a,b)-biregular bipartite graph with $|V(G)| \geq 2(a+b)$ and $G \in \mathfrak{N}$, then*

$$W(G) \leq |V(G)| - 3.$$

For general graphs, Kamalian proved the following

Theorem 3 *[6]. If G is a connected graph and $G \in \mathfrak{N}$, then*

$$W(G) \leq 2|V(G)| - 3.$$

The upper bound of Theorem 3 was improved in [5].

Theorem 4 *[5]. If G is a connected graph with $|V(G)| \geq 3$ and $G \in \mathfrak{N}$, then*

$$W(G) \leq 2|V(G)| - 4.$$

On the other hand, in [7] Petrosyan proved the following theorem.

Theorem 5 *For any $\varepsilon > 0$, there is a graph G such that $G \in \mathfrak{N}$ and*

$$W\left(G\right) \geq \left(2 - \varepsilon\right)\left|V\left(G\right)\right|.$$

For planar graphs, the upper bound of Theorem 3 was improved in [4].

Theorem 6 *[4]. If G is a connected planar graph and $G \in \mathfrak{N}$, then*

$$W(G) \leq \tfrac{11}{6}|V(G)|.$$

In this note we give a short proof of Theorem 3 based on Theorem 1. We also derive a new upper bound for the greatest possible number of colors in interval edge-colorings of regular graphs.

2. Main results

Proof of Theorem 3. Let $V(G)=\{v_1,v_2,\dots,v_n\}$ and α be an interval $W(G)$-coloring of the graph G. Define an auxiliary graph H as follows:

$$V(H)=U\cup W, \text{ where}$$

$$U=\{u_1,u_2,\dots,u_n\},\ W=\{w_1,w_2,\dots,w_n\} \text{ and}$$

$$E(H)=\{u_iw_j,u_jw_i|\ v_iv_j\in E(G),1\leq i\leq n,1\leq j\leq n\}\cup\{u_iw_i|\ 1\leq i\leq n\}.$$

Clearly, H is a connected bipartite graph with $|V(H)|=2|V(G)|$.

Define an edge-coloring β of the graph H in the following way:

(1) $\beta(u_iw_j)=\beta(u_jw_i)=\alpha(v_iv_j)+1$ for every edge $v_iv_j\in E(G)$,

(2) $\beta(u_iw_i)=\max S(v_i,\alpha)+2$ for $i=1,2,\dots,n$.

It is easy to see that β is an edge-coloring of the graph H with colors $2,3,\dots,W(G)+2$ and $\min S(u_i,\beta)=\min S(w_i,\beta)$ for $i=1,2,\dots,n$. Now we present an interval $(W(G)+2)$-coloring of the graph H. For that we take one edge $u_{i_0}w_{i_0}$ with $\min S(u_{i_0},\beta)=\min S(w_{i_0},\beta)=2$, and recolor it with color 1. Clearly, such a coloring is an interval $(W(G)+2)$-coloring of the graph H. Since H is a connected bipartite graph and $H\in\mathfrak{N}$, by Theorem 1, we have

$$W(G)+2\leq|V(H)|-1=2|V(G)|-1, \text{ thus}$$

$$W(G)\leq 2|V(G)|-3.$$

□

Theorem 7 *If G is a connected r-regular graph with $|V(G)|\geq 2r+2$ and $G\in\mathfrak{N}$, then*

$$W(G)\leq 2|V(G)|-5.$$

Proof. In a similar way as in the prove of Theorem 3, we can construct an auxiliary graph H and to show that this graph has an interval $(W(G)+2)$-coloring. Next, since H is a connected $(r+1)$-regular bipartite graph with $|V(H)|\geq 2(2r+2)$ and $H\in\mathfrak{N}$, by Theorem 2, we have

$$W(G)+2\leq|V(H)|-3=2|V(G)|-3, \text{ thus}$$

$$W(G)\leq 2|V(G)|-5.$$

□

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